Localization Properties of Two Interacting Electrons in a Disordered Quasi One–Dimensional Potential

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Abstract

We study the transport properties of two electrons in a quasi one–dimensional disordered wire. The electrons are subject to both, a disorder potential and a short range two–body interaction. Using the approach developed by Iida *et al.* [Ann. Phys. (N.Y.) **200** (1990) 219], the supersymmetry technique, and a suitable truncation of Hilbert space, we work out the two–point correlation function in the framework of a non–linear σ model. We study the loop corrections to arbitrary order. We obtain a remarkably simple and physically transparent expression for the change of the localization length caused by the two-body interaction.

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Non–interacting electrons in a one–dimensional random potential are localized. The presence of a short–range two–body interaction (a screened

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Coulomb potential) affects the localization length. For two interacting electrons in one dimension, this fact has been clearly established in a series of numerical and analytical investigations (see Refs. [1],[2],[3] and references therein). The localization length increases independently of the sign of the interaction when the electrons move together at short distance [3].

In the present work we address the more realistic and more demanding case of two electrons in a quasi one–dimensional disordered wire interacting via a short–range interaction. Using an analytic method developed recently [4], a reduction of Hilbert space and supersymmetry, we achieve a complete analytical solution of the problem. We show that the two–body interaction affects the localization length via the level density. We establish a criterion for the onset of interaction–induced effects on the localization length. Our final analytical expression allows for the numerical determination of the localization length for any given two–body interaction. To the best of our knowledge, this is the first time that an analytical solution of a problem involving both, disorder and interaction, has been achieved.

Model. We consider a quasi one–dimensional wire of length L which consists [5] of K slices labelled $a, b, c, \ldots, K$. The surfaces connecting neighboring slices are transverse to the axis of the wire. Eventually we take the limit where $K \to \infty$ and where the longitudinal extension of the slices goes to zero. In each slice we use an arbitrary basis of single–particle states labelled $|aj\rangle$ with $j = 1, 2, \ldots, l$ and Fermionic creation and annihilation operators $\alpha^\dagger_{aj}$ and α_{aj}, respectively. We later take the limit $l \to \infty$.

The Hamiltonian H is the sum of three terms, $H = H_0 + H_1 + H_2$. In every slice, disorder is simulated by an ensemble of random single–particle Hamiltonians belonging to the unitary ensemble,

$$H_0 = \sum_{aij} h^{(a)}_{ij} \alpha^\dagger_{ai} \alpha_{aj} \ . \tag{1}$$

The complex random variables $h^{(a)}_{ij}$ have a Gaussian distribution with mean value zero and the following non–vanishing second moments,

$$\overline{h^{(a)}_{ij} h^{(a')*}_{i'j'}} = \frac{\lambda^2}{l} \delta_{aa'} \delta_{ii'} \delta_{jj'} \ . \tag{2}$$

The overbar denotes the ensemble average and λ has the dimension of energy. The disorder Hamiltonians in different slices are uncorrelated. Hopping

between neighboring slices is described by the hopping term

$$H_1 = \sum_{ai} v[\alpha^\dagger_{ai}\alpha_{a+1i} + h.c.] \ . \tag{3}$$

The short–range two–body interaction acts only between electrons in the same slice and reads

$$H_2 = \sum_a \sum_{i<j,i'<j'} w_{iji'j'}\alpha^\dagger_{ai}\alpha^\dagger_{aj}\alpha_{aj'}\alpha_{ai'} \ . \tag{4}$$

The two–body matrix elements are antisymmetric in the pairs (ij) and $(i'j')$ and Hermitean. Both v and the $w_{iji'j'}$'s are non–random quantities independent of the slice label a. For the single–electron problem, it has been shown via supersymmetry [5] that H_0+H_1 defines the same non–linear sigma model as obtained from a continuum model using a kinetic term and a random potential. This is why our model is generic.

The Hilbert space of the two–electron problem is spanned by the orthonormal states $|ab\mu\rangle$ where $a \leq b$ and where μ stands for the pair (i,j) with $i<j$ for $a=b$. For $a<b$ fixed, the number N_{ab} of states is l^2 while for $a=b$, $N_{aa}=l(l-1)/2$. The totality of states $|ab\mu\rangle$ with $\mu=1,\ldots,N_{ab}$ is referred to as the box (ab). The effect of the two–body interaction on localization properties in the one–dimensional case is strongest when the two electrons stay close to each other [3]. This fact, and the need to reduce the dimension of Hilbert space, cause us to consider only the $2K-1$ boxes where the two electrons are either located in the same slice a (this is box (aa)), or in two adjacent slices a and $a+1$ (this is box $(a(a+1))$), with $a=1,\ldots,K$.

Supersymmetry. Localization properties can be read off the two–point correlation function

$$C(n) = \overline{|\langle a(a+1)\mu|(E^+-H)^{-1}|(a+n)(a+n+1)\nu\rangle|^2} \ . \tag{5}$$

To be independent of edge effects, we choose $1 \ll a \ll (a+n) \ll K$. For large n, $C(n)$ should decay exponentially in n. In proper units, the coefficient in the exponent defines the localization length. We calculate $C(n)$ using supersymmetry [6, 7].

Because of the Gaussian distribution of the coefficients $h^{(a)}_{ij}$, the ensemble average in Eq. (5) is entirely determined by the second moments of the matrix elements $\langle ab\mu|H_0|ab\sigma\rangle$ of H_0 with $b=a,(a+1)$, i.e., by the quantities

$$\mathcal{A}^{(k)}_{\mu\nu;\rho\sigma} = \frac{l}{\lambda^2}\overline{\langle ab\mu|H_0|ab\sigma\rangle\langle a'b'\rho|H_0|a'b'\nu\rangle} \ . \tag{6}$$

The index k labels the various possible combinations of (a, b, a', b'). In the direct product space $\{\mu\nu\}$, the matrix $\mathcal{A}^{(k)}$ can be diagonalized and expanded into bilinear forms of its eigenvectors [4],

$$\mathcal{A}^{(k)}_{\mu\nu;\rho\sigma} = \frac{1}{N(k)} \sum_{s=0}^{1} \Lambda^{(s)}(k) \sum_{\tau} C^{(s\tau)}_{\mu\nu}(k;l) C^{(s\tau)}_{\rho\sigma}(k;r)) \ , \tag{7}$$

where the $\Lambda^{(s)}(k)$ are real positive eigenvalues, $C^{(s\tau)}$ the corresponding right (r) and left (l) eigenvectors, τ distinguishes degenerate eigenvectors, and $N(k) \gg 1$ is a suitably chosen normalization factor related to N_{aa} and $N_{a(a+1)}$.

The generating function Z carries in the exponent terms of the form

$$-\frac{i}{2} \sum_{ab} \sum_{\mu\nu} \Psi^*_{ab\mu} L^{1/2} \langle ab\mu | H_0 | ab\nu \rangle L^{1/2} \Psi_{ab\nu} \ . \tag{8}$$

Here $\Psi_{ab\mu}$ is a graded vector. The graded matrix L is defined in Ref. [7]. After averaging Z, the quartic terms in the Ψ's appearing in the exponent involve the matrices $\mathcal{A}^{(k)}$. The eigenvalue expansion (7) allows us to use the Hubbard–Stratonovich (H–S) transformation. We introduce the matrices

$$\begin{aligned} A^{(aba'b';\tau)}_{\alpha\beta}(p) &= i\lambda \sum_{\mu\nu} L^{1/2}_{\alpha\alpha} \Psi_{ab\nu\alpha} \Psi^*_{a'b'\mu\beta} L^{1/2}_{\beta\beta} \\ &\quad \times C^{1\tau}_{\mu\nu}(aba'b';p) \end{aligned} \tag{9}$$

where p specifies whether a right– or left–hand eigenvector C is involved. The terms quartic in the Ψ's are bilinear (rather than quadratic) in the matrices A. This fact necessitates the introduction of suitable linear combinations of the A's. In the H–S tranformation, each such linear combination generates a σ-field. After integration over the Ψ's, the ensemble average of Z takes the form

$$\begin{aligned} \overline{Z} = \int \mathrm{d}[\sigma] \exp\Big[-\sum_k \mathrm{trg}\Big(& N(k)(\sigma^{(k)})^2 \\ & - \mathrm{trg\ tr\ ln\ } \mathbf{N}(J) \Big) \Big] \ . \end{aligned} \tag{10}$$

The sum over k extends over all σ-fields, the volume element is denoted by $\mathrm{d}[\sigma]$. Here $\mathbf{N}(J)$ is both, a graded matrix and a matrix in Hilbert space with

basis vectors $|ab\mu\rangle$ and $b = a, (a+1)$. The symbol J denotes the source terms [7]. For lack of space, we give the structure of $\mathbf{N}(J)$ only for the cases $a = a', b = b'$ where

$$\langle aa\mu|\mathbf{N}(J)|\ aa\nu\rangle = \Big(E - \lambda\sigma^{(aa)}\Big)\delta_{\mu\nu} - \sum_{\tau} \lambda\sigma^{(aa;\tau)} C^{(1\tau)}_{\mu\nu}(0)\ - w_{\mu\nu} \tag{11}$$

and

$$\langle a(a+1)\mu|\mathbf{N}(J)|\ a(a+1)\nu\rangle = \Big(E - -\lambda\sigma^{(a(a+1))}\Big)\ \delta_{\mu\nu} - \sum_{\tau} \lambda\sigma^{(a(a+1);\tau)} C^{(1\tau)}_{\mu\nu}(1)\ . \tag{12}$$

Non–diagonal elements like $\langle (a-1)a\mu|\mathbf{N}(J)|aa\nu\rangle$ contain hopping contributions.

Saddle Points. The factors $N(k)$ in Eq. (10) obey $N(k) \gg 1$. Hence, we use the saddle–point approximation with $v = 0$ [5]. The saddle–point equations can be cast into the form of a Pastur equation,

$$\langle ab\mu|X|a'b'\nu\rangle = \sum_{\rho\sigma} \overline{\times\langle ab\mu|H_0|ab\sigma\rangle\Big(\frac{1}{E - w - X}\Big)_{(ab|a'b');\sigma\rho}\langle a'b'\rho|H_0|a'b'\nu\rangle}\ . \tag{13}$$

The matrix X contains all the σ–matrices which occur in $\mathbf{N}(J)$.

Eq. (14) implies that the solutions $\sigma^{(k)}_{\rm sp}$ are zero or approximately zero except for those with $k = (a, (a+1), a, (a+1))$ and $k = (a, a, a, a)$. The corresponding fields obey the equations

$$\sigma^{(a(a+1))}_{\rm sp} = \frac{\lambda}{E - \lambda\sigma^{(a(a+1))}_{\rm sp}} \tag{14}$$

and

$$\sigma^{(aa)}_{\rm sp} = N^{-1}_{aa}\ {\rm tr}[(\frac{\lambda}{E - w - \lambda\sigma^{(aa)}_{\rm sp}})_{\mu\nu}]\ . \tag{15}$$

The diagonal solution for $\sigma^{(a(a+1))}_{\rm sp}$ takes the standard form $\sigma^{(a(a+1))}_{\rm d} = (E/(2\lambda)) \pm i\ \Delta_1(E)$, with $\Delta_1(E) = \sqrt{1 - (E/(2\lambda))^2}$. Here $\Delta_1(E)$ is proportional to the

spectral density $\rho_{\rm sp1}(E)$ in box $(a(a+1))$ and has the shape of a semicircle. This fact points to a deficiency of the saddle–point approximation: The two electrons in slices a and $(a+1)$ do not interact. The spectrum of each has semicircular shape. The spectrum in box $(a(a+1))$ is, thus, the convolution of two semicircles and not a semicircle. This shows that we must take into account loop corrections to the saddle–point approximation. This is done below. The invariance of the saddle–point equation Eq. (14) under pseudounitary graded transformations T implies that the saddle–point manifold has the form [7]

$$\sigma_{\rm sp}^{(a(a+1))} = T^{(a(a+1))} \ [\ \frac{E}{2\lambda} - i \ \Delta_1(E) \ L \ ] \ (T^{(a(a+1))})^{-1} \ . \tag{16}$$

Eq. (15) for $\sigma_{\rm sp}^{(aa)}$ contains the two-body interaction w. Diagonalizing w and using a simple geometrical construction shows that Eq. (15) possesses either N_{aa} real solutions (this applies for $E < E_1$ and for $E > E_2$), or, for $E_1 < E < E_2$, $(N_{aa} - 2)$ real and two complex conjugate solutions $a(E) \pm i\Delta_0(E)$. Thus,

$$\sigma_{\rm sp}^{(aa)} = T^{(aa)} \ [\ a(E) - i\Delta_0(E) \ L \ ] \ (T^{(aa)})^{-1} \ , \tag{17}$$

where $\Delta_0(E) > 0$ is proportional to the spectral density $\rho_{\rm sp0}(E)$ in box (aa). The essential difference between the saddle–point solutions in Eqs. (16) and (17) lies in the difference between $\Delta_0(E)$ and $\Delta_1(E)$, i.e, in the different spectral densities $\rho_{\rm sp0}(E)$ and $\rho_{\rm sp1}(E)$. The interaction w deforms the semi–circle $\Delta_1(E)$. Aside from a possible overall shift of the spectrum, w causes $\Delta_0(E)$ to be smaller in the centre of the semicircle, and to extend beyond the end points $(-2\lambda, +2\lambda)$ of $\Delta_1(E)$. For a weak interaction, we calculate $E_{1,2}$ using a second–order perturbation expansion in w which yields

$$E_{1,2} = E_0 \pm (2 + U^2)\lambda \ . \tag{18}$$

Here $E_0 = (1/N_{aa}){\rm tr}(w)$ and $U^2 = \{(1/N_{aa}){\rm tr}(w^2) - [(1/N_{aa}){\rm tr}(w)]^2\}/\lambda^2$. This shows shift and widening of the spectrum. If we identify the matrix elements $h_{ij}^{(a)}$ with those of the impurity potential $V_{\rm imp}$, it is easy to see that a qualitative change of the spectral density occurs whenever

$$\langle aa\mu|w^2|aa\mu\rangle_{\rm av} \geq \langle ai|(V_{\rm imp})^2|ai\rangle_{\rm av} \tag{19}$$

where the matrix elements are averaged over μ and i, respectively. This is the announced qualitative criterion.

Localization. To calculate localization properties, we expand the effective Lagrangian in the exponent of Eq. (10) in powers of v, keeping terms up to second order. We put $J = 0$ and use that for $k = (a, a, a, a)$ and $k = (a, a+1, a, a+1)$, the $\sigma_{\rm sp}^{(k)}$'s are solutions of the saddle–point Eqs. (14) and (15). We also use Eqs. (16) and (17). The terms quadratic in the $\sigma_{\rm sp}^{(k)}$'s vanish, and the matrix $\mathbf{N}$ yields

$$+(v/\lambda)^2(3l^2/4)\Delta_0\Delta_1 \sum_{j=1}^{K'} {\rm trg}\Big(T^{(j)} L (T^{(j)})^{-1} \times T^{(j+1)} L (T^{(j+1)})^{-1} \Big) \tag{20}$$

where the $2K-1$ boxes (aa) and $a(a+1)$ of the two–electron problem have been mapped onto $K' = 2K-1$ slices by putting $j = 2a-1$ for the boxes (aa) and $j = 2a$ for the boxes $(a(a+1)$. The expression (20) has exactly the form of the non–linear sigma model derived for the transport of a single electron through a disordered wire in Ref. [5]. Since Δ_0 and Δ_1 are proportional to the spectral densities $\rho_{\rm sp0}(E)$ and $\rho_{\rm sp1}(E)$ in boxes (aa) and $(a(a+1))$, respectively, as determined by the saddle–point condition, Eq. (20) suggests defining the saddle–point approximation $T_{\rm sp}$ to the transport coefficient by

$$T_{\rm sp} = 2\pi \ \rho_{\rm sp0}(E) \ v^2 \ \rho_{\rm sp1}(E) \ . \tag{21}$$

The localization length is proportional to $T_{\rm sp}$. We conclude that the ratio of the localization lengths $\zeta(w \neq 0)$ for non–vanishing two–body interaction and $\zeta(w = 0)$ for vanishing two–body interaction is given by

$$\frac{\zeta(w \neq 0)}{\zeta(w = 0)} = \frac{\rho_{\rm sp0}(E, w \neq 0)}{\rho_{\rm sp0}(E, w = 0)} \ . \tag{22}$$

This result shows that and how the two-body interaction modifies the localization length. Depending upon the energy considered and on the sizes of the shift and of the widening of the spectrum in box (aa) compared to the unperturbed spectrum in box $(a(a+1))$, the localization length may decrease or increase. In general, we expect an increase near the edge of the unperturbed spectrum, and a decrease in the center.

Loop Corrections. The result Eq. (22) is obtained from the saddle–point approximation, and it was remarked earlier that this approximation is deficient. Therefore, we now study the loop corrections to the saddle–point solution. We do so by expanding the σ's around the saddle–point solutions $\sigma_{\rm sp}^{(k)}$. We recall that some of these vanish. We write

$$\sigma^{(k)} = \sigma_{\rm sp}^{(k)} + T^{(k)} \delta\sigma^{(k)} (T^{(k)})^{-1} \ . \tag{23}$$

We expand $\overline{Z}$ in powers of the $\delta\sigma^{(k)}$'s, keeping in the exponent only the quadratic terms which originate from the corresponding terms in Eq. (10). We need to address the form of the source terms J contained in $\mathbf{N}(J)$. The expression (5) is reproduced if the source term has two non–zero matrix elements, $\langle a(a+1)\mu|J|(a+n)(a+n+1)\nu\rangle$ and $\langle (a+n)(a+n+1)\nu|J|a(a+1)\mu\rangle$. The loop expansion generates contributions in which, besides the $\delta\sigma^{(k)}$'s, powers of J and v appear to all orders. We clearly need consider only terms of second order in J. We first consider such terms which are of zeroth order in v. Inspection shows that, for $n \gg 1$ and to arbitrary order in the $\delta\sigma^{(k)}$'s, the result Eq. (22) remains valid after Wick contraction of the $\delta\sigma^{(k)}$'s. Terms of order v^2 come in two classes. There are corrections of order v^2 which either appear under the same trace(s) as the source terms to begin with, or which become connected to the source terms through Wick contractions of the $\delta\sigma$'s, and there are other such corrections for which this is not the case. Formally, the latter can be calculated by putting the source terms equal to zero. We can show that after Wick contraction, every one of these latter terms takes the form

$$\begin{aligned} & c\ v^2 \sum_{j=1}^{K'} {\rm trg}\Big(T^{(j)} L (T^{(j)})^{-1} \\ & \times T^{(j+1)} L (T^{(j+1)})^{-1} \Big) \ , \end{aligned} \tag{24}$$

with c a constant which depends on the particular term considered. Re–exponentiation of this result yields a renormalization of the transport coefficient $T_{\rm sp}$. This is a very satisfying result. Although we are not able to calculate the values of the coefficients c analytically, we can show that each of these coefficients is determined entirely by two neighboring boxes, say (aa) and $(a(a+1))$. Any such pair of boxes yields the same result. Thus, our analytical approach has reduced the calculation of the renormalized transport

coefficient to a problem involving only two neighboring boxes, and not the full Hamiltonian H. To understand the physical origin and meaning of the renormalization terms in this reduced framework, we use the diagrammatic expansion of Ref. [8] which is akin to a diagrammatic impurity perturbation expansion. One can see heuristically that the correction terms add up to yield the ensemble average of the product of the physical level densities $\rho_0(E)$ and $\rho_1(E)$ in the two boxes. Thus, the sum of the saddle–point value and of all the loop corrections yields for the renormalized transport coefficient T the expression

$$T(w) = 2\pi \ v^2 \ \overline{\rho_0(E,w)\rho_1(E)} \ . \tag{25}$$

The ensemble average in Eq. (25) can be determined without difficulty by numerical simulation for every given two–body interaction w. We must finally take the continuum limit by letting the longitudinal length d of each slice go to zero and the number K of slices go to infinity. In this limit we have

$$\frac{\zeta(w \neq 0)}{\zeta(w=0)} = \lim_{d\to 0} \frac{T(w \neq 0)}{T(w=0)} \ . \tag{26}$$

We believe that in a numerical simulation, the ratio of localization lengths will be fairly independent of d.

Summary. We have studied the localization properties of two interacting electrons in a quasi one–dimensional wire. Both electrons move in a disorder potential and interact when they are close to each other. Hopping allows the electrons to move along the wire.

Our central result is contained in Eqs. (25,26). The two–body interaction affects the localization length because it alters the level density in the boxes containing two electrons. Depending on the energy chosen and on the sign of the two–body interaction, the localization length may decrease or increase. To the best of our knowledge, this is the first time that an analytical treatment of both, disorder and interaction, has been possible.

The supersymmetry method does not apply to one–dimensional systems. Hence it is difficult to compare our results with previous numerical work. The exception is the work of Imry [2] who derives an expression (his Eq. (4)) for the change of the localization length which applies also in higher dimensions and is proportional to $(U/B)^2$, with U a typical two–body matrix element and B the bandwidth. This expression differs from our expression (25) in which we retain the structure typical for the Thouless block scaling argument

with hopping. We predict a change of localization length which depends upon all moments $\mathrm{tr}(w), \mathrm{tr}(w^2), \ldots$ of the two–body interaction while Imry's result involves only his U^2 and is, thus, independent of the sign of w.

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